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Additivity of multiplicative maps on triangular rings ☆

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In this paper we shall give a unified technique in the discussion of the additivity of n-multiplicative automorphisms, n-multiplicative derivations, n-elementary surjective maps, and Jordan multiplicative surjective maps on triangular rings.

1. Introduction

Let R and R' be associative rings (not necessarily with the identity elements). We denote by $Z(R)$ the center of R. Set $a \circ b = ab + ba$.

A bijective map σ of R onto R' is called a *n-multiplicative isomorphism* if

$$(x_1 x_2 \cdots x_n)^\sigma = x_1^\sigma x_2^\sigma \cdots x_n^\sigma \quad \text{for all } x_1, x_2, \ldots, x_n \in R.$$

In particular, if $n = 2$, then σ is called a *multiplicative isomorphism*. Similarly, a map d of R is called a *n-multiplicative derivation* if

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$$(x_1 \cdots x_n)^d = \sum_{i=1}^{n} x_1 \cdots x_i^d \cdots x_n \quad \text{for all } x_1, x_2, \ldots, x_n \in R.$$

If $(xy)^d = x^d y + xy^d$ for all $x, y \in R$, then d is called a *multiplicative derivation*.

Let $M : R \to R'$ and $M^* : R' \to R$ be two maps. Call the ordered pair (M, M^*) a *n-elementary map* of $R \times R'$ if M and M^* satisfy

$$M(x_1 M^*(y_1) z_1 \cdots x_n M^*(y_n) z_n) = M(x_1) y_1 M(z_1) \cdots M(x_n) y_n M(z_n)$$

and

$$M^*(y_1 M(x_1) u_1 \cdots y_n M(x_n) u_n) = M^*(y_1) x_1 M^*(u_1) \cdots M^*(y_n) x_n M^*(u_n)$$

for all $x_1, \ldots x_n, z_1, \ldots, z_n \in R, y_1, \ldots, y_n, u_1, \ldots, u_n \in R'$. In particular, if $n = 1$, then (M, M^*) is the well-known elementary map of $R \times R'$ introduced by Brešar and Šemrl [1].

Call the ordered pair (M, M^*) a *Jordan map* of $R \times R'$ if

$$\begin{aligned} &M(aM^*(x) + M^*(x)a) = M(a)x + xM(a), \\ &M^*(M(a)x + xM(a)) = aM^*(x) + M^*(x)a \end{aligned}$$

for all $a \in R, x \in R'$. Obviously, if ϕ is a Jordan isomorphism of R into R', then the ordered pair (ϕ, ϕ^{-1}) is a Jordan map of $R \times R'$.

It is an interesting problem to study the interrelation between the multiplicative and the additive structure of a ring. It is Martindale who first established a condition on a ring R such that every multiplicative isomorphism on R is additive [12, Theorem]. The problem has been studied in rings and other algebraic systems (see [2,4–11,13–16]).

In this paper we shall discuss the additivity of multiplicative maps in a more general setting than a triangular algebra (see [3] for its definition). More precisely, we give the following definition.

Definition 1.1. Let A and B be two rings and let M be an (A, B)-bimodule such that

(i) M is faithful as a left A-module and faithful as a right B-module,
(ii) if $m \in M$ is such that $AmB = 0$ then $m = 0$.

The ring

$$\mathrm{Tri}(A, M, B) = \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix} | a \in A, \ m \in M, b \in B \right\}$$

under the usual matrix addition and formal matrix multiplication will be called a triangular ring.

Let $T = \mathrm{Tri}(A, M, B)$. We denote by $a \oplus b$ the element

$$\begin{pmatrix} a & 0 \\ & b \end{pmatrix}$$

in T.

We borrow the ideas of [12]. Set

$$T_{11} = \left\{ \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix} | a \in A \right\}, \quad T_{12} = \left\{ \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} | m \in M \right\},$$

and

$$T_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix} | b \in B \right\}.$$

Then we can write $T = T_{11} \bigoplus T_{12} \bigoplus T_{22}$. In what follows, when we write a_{ij}, it indicates $a_{ij} \in T_{ij}$ and the corresponding elements in A, B, or M. Note that $a_{ij} a_{kl} = 0$ if $j \neq k$.

We define two natural projections $\pi_A : T \to A$ and $\pi_B : T \to B$ by

$$\pi_A : \begin{pmatrix} a & m \\ & b \end{pmatrix} \mapsto a \quad \text{and} \quad \pi_B : \begin{pmatrix} a & m \\ & b \end{pmatrix} \mapsto b.$$

Proposition 1.1. *Let $T = Tri(A, M, B)$ be a triangular ring. The center of T is*

$$Z(T) = \{a \oplus b | am = mb \quad \text{for all } m \in M\}.$$

Furthermore, $Z(T)_{11} \cong \pi_A(Z(T)) \subseteq Z(A)$ and $Z(T)_{22} \cong \pi_B(Z(T)) \subseteq Z(B)$, and there exists a unique ring isomorphism τ from $\pi_A(Z(T))$ to $\pi_B(Z(T))$ such that $am = m\tau(a)$ for all $m \in M$.

Proof. Let $S = \{a \oplus b | am = mb$ for all $m \in M\}$. It is easy to check that if $a \in Z(A)$, $b \in Z(B)$ and $am = mb$ for every $m \in M$, then $a \oplus b \in Z(T)$; that is, $(Z(A) \oplus Z(B)) \cap S \subseteq Z(T)$. To prove that $S = Z(T)$, it suffices to show that $Z(T) \subseteq S$ and $S \subseteq Z(A) \oplus Z(B)$.

Suppose that

$$x = \begin{pmatrix} a & n \\ & b \end{pmatrix} \in Z(T).$$

By the fact that $x(a' \oplus 0) = (a' \oplus 0)x$ for all $a' \in A$, we have $a'n = 0$ for all $a' \in A$ and so $n = 0$. For any $m \in M$, we have

$$\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} x = x \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}$$

and hence $am = mb$. Therefore, $Z(T) \subseteq S$.

Suppose that $a \oplus b \in S$. Then for any $a' \in A$ we have

$$(aa' - a'a)m = a(a'm) - a'(am) = (a'm)b - a'(mb) = 0 \quad \text{for all } m \in M,$$

and hence $aa' - a'a = 0$ as M is left faithful A-module. Therefore $a \in Z(A)$, and similarly we have $b \in Z(B)$. Hence, $S \subseteq Z(A) \oplus Z(B)$.

The fact that $\pi_A(Z(T)) \subseteq Z(A)$ and $\pi_B(Z(T)) \subseteq Z(B)$ are direct consequences of $Z(T) = S \subseteq Z(A) \oplus Z(B)$. It remains to prove the existence of the ring isomorphism $\tau : \pi_A(Z(T)) \to \pi_B(Z(T))$.

Take $a \oplus b, a \oplus b' \in Z(T)$. For any $m \in M$, we have $mb = am = mb'$. Since M is right faithful B-module, $b = b'$. Therefore, for any $a \in \pi_A(Z(T))$, there exists a unique $b \in \pi_B(Z(T))$, denoted by $\tau(a)$, such that $a \oplus b \in Z(T)$. It is easy to see that τ is also surjective. It remains to prove that τ is a ring isomorphism.

For any $a, a' \in \pi_A(Z(T))$, we have

$$\begin{aligned} (a + a')m &= m(\tau(a) + \tau(a')), \\ (aa')m &= a(a'm) = (a'm)\tau(a) = a'(m\tau(a)) = m\tau(a)\tau(a'), \end{aligned}$$

thus $\tau(a + a') = \tau(a) + \tau(a')$ and $\tau(aa') = \tau(a)\tau(a')$, proving τ is a ring isomorphism. □

Proposition 1.2. *Let $T = Tri(A, M, B)$ be a triangular ring such that*

(*i*) *For $a \in A$, if $aA = 0$, then $a = 0$;*
(*ii*) *For $b \in B$, if $Bb = 0$, then $b = 0$.*

For $u \in T$, if $uT = 0$ or $Tu = 0$, then $u = 0$.

Proof. Suppose that $Aa = 0$. We claim that $a = 0$. Indeed, for every $m \in M$ we have $AamB = 0$ and so $am = 0$ by condition (ii) in Definition 1.1. That is, $aM = 0$. Hence $a = 0$. Similarly, if $bB = 0$ then $b = 0$.

Suppose that $uT = 0$. Write $u = u_{11} + u_{12} + u_{22}$, where $u_{ij} \in T_{ij}$. In particular, $u_{11}A = 0$, $u_{22}B = 0$, and $u_{12}B = 0$. By assumption we obtain that $u_{11} = 0$ and $u_{12} = 0$. While $u_{22}B = 0$ implies that $u_{22} = 0$. Hence $u = 0$. The case $Tu = 0$ can be proved analogously. □

2. The main result and two corollaries

For the discussion of the additivity of multiplicative maps in triangular rings we shall prove the following main result of this paper. The basic idea comes from [16].

Theorem 2.1. *Let $B : T \times T \to T$ be a biadditive map such that*

(1) $B(T_{ii}, T_{jj}) \subseteq T_{ii} \cap T_{jj}, B(T_{ii}, T_{12}) \subseteq T_{12}, B(T_{12}, T_{ii}) \subseteq T_{12}, B(T_{12}, T_{12}) = 0$;
(2) $B(a_{12}, T_{22}) = 0$ *or* $B(T_{11}, a_{12}) = 0$ *implies* $a_{12} = 0$;
(3) $B(T_{22}, a_{22}) = 0$ *implies* $a_{22} = 0$;
(4) $B(a_{11} + a_{22}, T_{12}) = B(T_{12}, a_{11} + a_{22}) = 0$ *implies that* $a_{11} \oplus (-a_{22}) \in Z(T)$;
(5) *If* $c_{11} \oplus c_{22} \in Z(T)$ *then* $B(c_{11}, a_{11}) = B(a_{11}, c_{11}), B(c_{22}, a_{22}) = B(a_{22}, c_{22})$, *and* $B(c_{11}, a_{11})a_{12}a_{22} = a_{11}a_{12}B(c_{22}, a_{22})$;
(6) $B(a_{11}, B(a_{12}, a_{22})) = B(B(a_{11}, a_{12}), a_{22})$.

If a map $f : T \times T \to T$ satisfies

(i) $f(T, 0) = f(0, T) = 0$;
(ii) $B(f(x, y), z) = f(B(x, z), B(y, z))$;
(iii) $B(z, f(x, y)) = f(B(z, x), B(z, y))$

for all $x, y, z \in T$, then $f = 0$.

Proof. For every $x_{ij}, u_{ij}, a_{ij} \in T_{ij}$ we divide the proof into the following four steps.

Step 1. By assumption we have

$$B(f(x_{11}, x_{12}), a_{22}) = f(B(x_{11}, a_{22}), B(x_{12}, a_{22})) = f(0, B(x_{12}, a_{22})) = 0.$$

Then

$$B(f(x_{11}, x_{12})_{11}, a_{22}) + B(f(x_{11}, x_{12})_{12}, a_{22}) + B(f(x_{11}, x_{12})_{22}, a_{22}) = 0.$$

By assumption (1) we see that $B(f_{11}, x_{12})_{11}, a_{22}) = 0, B(f(x_{11}, x_{12})_{12}, a_{22}) \in T_{12}$, and $B(f(x_{11}, x_{12})_{22}, a_{22}) \in T_{22}$. We get from the above identity that

$$B(f(x_{11}, x_{12})_{12}, a_{22}) = 0.$$

Hence $f(x_{11}, x_{12})_{12} = 0$ by assumption (2). Since

$$B(a_{22}, f(x_{11}, x_{12})) = f(B(a_{22}, x_{11}), B(a_{22}, x_{12})) = f(0, B(a_{22}, x_{12})) = 0,$$

we get $B(a_{22}, f(x_{11}, x_{12})_{22}) = 0$ by assumption (1) and so $f(x_{11}, x_{12})_{22} = 0$ by assumption (3). It follows that

$$B(f(x_{11}, x_{12})_{11}, a_{12}) = B(f(x_{11}, x_{12}), a_{12}) = f(B(x_{11}, a_{12}), 0) = 0,$$

and

$$B(a_{12}, f(x_{11}, x_{12})_{11}) = B(a_{12}, f(x_{11}, x_{12})) = f(B(a_{12}, x_{11}), 0) = 0.$$

Hence $f(x_{11}, x_{12})_{11} \oplus 0 \in Z(T)$ by assumption (4) and so $f(x_{11}, x_{12})_{11} = 0$ by Proposition 1.1. Consequently $f(T_{11}, T_{12}) = 0$.

Step 2. By assumption we have

$$B(f(x_{12}, u_{12}), a_{12}) = f(B(x_{12}, a_{12}), B(u_{12}, a_{12})) = f(0, 0) = 0.$$

That is,

$$B(f(x_{12}, u_{12})_{11}, a_{12}) + B(f(x_{12}, u_{12})_{12}, a_{12}) + B(f(x_{12}, u_{12})_{22}, a_{12}) = 0.$$

Since $B(T_{12}, T_{12}) = 0$ we get from the last identity that

$$B(f(x_{12}, u_{12})_{11} + f(x_{12}, u_{12})_{22}, a_{12}) = 0.$$

Similarly, we have $B(a_{12}, f(x_{12}, u_{12})_{11} + f(x_{12}, u_{12})_{22}) = 0$. It follows from assumption (4) that

$$f(x_{12}, u_{12})_{11} \oplus (-f(x_{12}, u_{12})_{22}) \in Z(T).$$

Since $B(T_{12}, T_{22}) \subseteq T_{12}$ and

$$B(f(x_{12}, u_{12}), a_{22}) = f(B(x_{12}, a_{22}), B(u_{12}, a_{22}))$$

we get from the above discussion that

$$B(f(x_{12}, u_{12}), a_{22})_{11} \oplus (-B(f(x_{12}, u_{12}), a_{22})_{22}) \in Z(T).$$

Since $B(f(x_{12}, u_{12}), a_{22}) \in T_{12} + T_{22}$ we have $B(f(x_{12}, u_{12}), a_{22})_{11} = 0$ and so $B(f(x_{12}, u_{12}), a_{22})_{22} = 0$ by Proposition 1.1. Hence $B(f(x_{12}, u_{12}), a_{22}) \in T_{12}$.

Next, by assumption (6) and Step 1 we have

$$\begin{aligned} B(a_{11}, B(f(x_{12}, u_{12}), a_{22})) &= B(a_{11}, f(B(x_{12}, a_{22}), B(u_{12}, a_{22}))) \\ &= f(B(a_{11}, B(x_{12}, a_{22})), B(a_{11}, B(u_{12}, a_{22}))) \\ &= f(B(a_{11}, a_{22} + B(x_{12}, a_{22})), B(B(a_{11}, u_{12}), a_{22})) \\ &= f(B(a_{11}, a_{22} + B(x_{12}, a_{22})), B(B(a_{11}, u_{12}), a_{22} + B(x_{12}, a_{22}))) \\ &= B(f(a_{11}, B(a_{11}, u_{12})), a_{22} + B(x_{12}, a_{22})) \\ &= B(0, a_{22} + B(x_{12}, a_{22})) = 0. \end{aligned}$$

Then $B(f(x_{12}, u_{12}), a_{22}) = 0$ by assumption (2). That is,

$$B(f(x_{12}, u_{12})_{11}, a_{22}) + B(f(x_{12}, u_{12})_{12}, a_{22}) + B(f(x_{12}, u_{12})_{22}, a_{22}) = 0.$$

Hence $B(f(x_{12}, u_{12})_{12}, a_{22}) = 0$ and so $f(x_{12}, u_{12})_{12} = 0$ by assumption (2). It follows that

$$B(a_{22}, f(x_{12}, u_{12})_{22}) = B(a_{22}, f(x_{12}, u_{12})) = f(B(a_{22}, x_{12}), B(a_{22}, u_{12})).$$

Since $B(T_{22}, T_{12}) \subseteq T_{12}$ we get from the above discussion that

$$B(a_{22}, f(x_{12}, u_{12})_{22})_{11} \oplus (-B(a_{22}, f(x_{12}, u_{12})_{22})_{22}) \in Z(T).$$

Since $B(a_{22}, f(x_{12}, u_{12})_{22}) \in T_{22}$ we have $B(a_{22}, f(x_{12}, u_{12})_{22})_{11} = 0$ and so

$$B(a_{22}, f(x_{12}, u_{12})_{22}) = B(a_{22}, f(x_{12}, u_{12})_{22})_{22} = 0,$$

making use of Proposition 1.1. This yields $f(x_{12}, u_{12})_{22} = 0$ by assumption (3) and consequently $f(x_{12}, u_{12})_{11} = 0$. Hence $f(T_{12}, T_{12}) = 0$.

Step 3. In view of Step 2 we have

$$\begin{aligned} B(f(x_{11} + x_{12}, u_{11} + u_{12}), a_{12}) &= f(B(x_{11} + x_{12}, a_{12}), B(u_{11} + u_{12}, a_{12})) \\ &= f(B(x_{11}, a_{12}), B(u_{11}, a_{12})) = 0. \end{aligned}$$

It implies that

$$B(f(x_{11} + x_{12}, u_{11} + u_{12})_{11} + f(x_{11} + x_{12}, u_{11} + u_{12})_{22}, a_{12}) = 0.$$

Similarly, we have

$$B(a_{12}, f(x_{11} + x_{12}, u_{11} + u_{12})_{11} + f(x_{11} + x_{12}, u_{11} + u_{12})_{22}) = 0.$$

It follows from assumption (4) that

$$f(x_{11} + x_{12}, u_{11} + u_{12})_{11} \oplus (-f(x_{11} + x_{12}, u_{11} + u_{12})_{22}) \in Z(T).$$

In view of Step 2 we have

$$\begin{aligned} B(f(x_{11} + x_{12}, u_{11} + u_{12}), a_{22}) &= f(B(x_{11} + x_{12}, a_{22}), B(u_{11} + u_{12}, a_{22})) \\ &= f(B(x_{12}, a_{22}), B(u_{12}, a_{22})) = 0. \end{aligned}$$

It further implies that

$$B(f(x_{11}+x_{12},u_{11}+u_{12})_{12},a_{22})=0.$$

Hence $f(x_{11}+x_{12},u_{11}+u_{12})_{12}=0$ by assumption (2).

In view of Step 2 we have

$$\begin{aligned}B(a_{22},f(x_{11}+x_{12},u_{11}+u_{12}))&=f(B(a_{22},x_{11}+x_{12}),B(a_{22},u_{11}+u_{12}))\\&=f(B(a_{22},x_{12}),B(a_{22},u_{12}))=0.\end{aligned}$$

This implies that

$$B(a_{22},f(x_{11}+x_{12},u_{11}+u_{12})_{22})=0.$$

Hence $f(x_{11}+x_{12},u_{11}+u_{12})_{22}=0$ by assumption (3). Consequently $f(x_{11}+x_{12},u_{11}+u_{12})_{11}=0$ as well. Therefore, $f(x_{11}+x_{12},u_{11}+u_{12})=0$.

Step 4. Since $B(T,T_{12})\subseteq T_{12}$ we have

$$B(f(x,u),a_{12})=f(B(x,a_{12}),B(u,a_{12}))=0,$$

making use of Step 2. So

$$B(f(x,u)_{11}+f(x,u)_{22},a_{12})=0.$$

Similarly, we have

$$B(a_{12},f(x,u)_{11}+f(x,u)_{22})=0.$$

It follows from assumption (4) that

$$f(x,u)_{11}\oplus(-f(x,u)_{22})\in Z(T).$$

Since $B(T_{11},T)\subseteq T_{11}+T_{12}$ we get

$$B(a_{11},f(x,u))=f(B(a_{11},x),B(a_{11},u))=0,$$

making use of Step 3. Consequently,

$$B(a_{11},f(x,u)_{11})+B(a_{11},f(x,u)_{12})=0.$$

It implies that $B(a_{11},f(x,u)_{11})=0$ and $B(a_{11},f(x,u)_{12})=0$. Now, the second identity implies $f(T,T)_{12}=0$ by assumption (2), while the first one implies

$$\begin{aligned}0=B(a_{11},f(x,u)_{11})a_{12}a_{22}&=B(f(x,u)_{11},a_{11})a_{12}a_{22}\\&=a_{11}a_{12}B(-f(x,u)_{22},a_{22})\\&=a_{11}a_{12}B(a_{22},-f(x,u)_{22}),\end{aligned}$$

making use of assumption (5). It follows that $B(a_{22},f(x,u)_{22})=0$ and so $f(T,T)_{22}=0$ by assumption (3). Hence, $f(T,T)_{11}=0$ as well. Thus, we conclude that $f=0$. □

Corollary 2.1. *Let $T=Tri(A,M,B)$ be a triangular ring such that*

(*i*) *For $a\in A$, if $aA=0$, then $a=0$;*
(*ii*) *For $b\in B$, if $Bb=0$, then $b=0$.*

Let k be a positive integer. If a map $f:T\times T\to T$ satisfies

(*i*) $f(T,0)=f(0,T)=0$;
(*ii*) $f(x,y)z_1z_2\cdots z_k=f(xz_1z_2\cdots z_k,yz_1z_2\cdots z_k)$;
(*iii*) $z_1z_2\cdots z_kf(x,y)=f(z_1z_2\cdots z_kx,z_1z_2\cdots z_ky)$,

for all $x,y,z_1,z_2,\ldots,z_k\in T$, then $f=0$.

Proof. We first claim that $f(x,y)z = f(xz, yz)$ and $zf(x,y) = f(zx, zy)$ for all $x, y, z \in T$. Indeed, since

$$f(x,y)(zz_1)z_2 \cdots z_k = f(xzz_1z_2 \cdots z_k, yzz_1z_2 \cdots z_k) = f(xz, yz)z_1z_2 \cdots z_k,$$

that is, $(f(x,y)z - f(xz,yz))T^k = 0$. Hence $f(x,y)z = f(xz,yz)$ by Proposition 1.2. Analogously, $zf(x,y) = f(zx, zy)$.

Define $B : T \times T \to T$ by

$$B(x,y) = xy.$$

It is easy to check that B and f satisfy the all conditions of Theorem 2.1. Hence $f = 0$. □

Corollary 2.2. *Let $T = Tri(A, M, B)$ be a triangular ring such that, for $b \in B$, $b \circ b' = 0$ for all $b' \in B$ implies $b = 0$. If a map $f : T \times T \to T$ satisfies*

(*i*) $f(T, 0) = f(0, T) = 0,$
(*ii*) $f(x,y) \circ z = f(x \circ z, y \circ z)$

for all $x, y, z \in T$, then $f = 0$.

Proof. Define $B : T \times T \to T$ by

$$B(x,y) = x \circ y.$$

It is easy to check that B and f satisfy the all conditions of Theorem 2.1. Hence $f = 0$. □

3. Applications

In this section we shall apply Corollary 2.1 and Corollary 2.2 to the discussion of the additivity of multiplicative maps on triangular rings.

Theorem 3.1. *Let $T = Tri(A, M, B)$ be a triangular ring such that*

(*i*) *For $a \in A$, if $aA = 0$, then $a = 0$;*
(*ii*) *For $b \in B$, if $Bb = 0$, then $b = 0$.*

Then every n-multiplicative isomorphism from T onto a ring R is additive.

Proof. Suppose that σ is a n-multiplicative isomorphism from T onto a ring R. Since σ is onto, $x^\sigma = 0$ for some $x \in T$. Then $0^\sigma = (0 \cdots 0x)^\sigma = 0^\sigma \cdots 0^\sigma x^\sigma = 0^\sigma \cdots 0^\sigma 0 = 0$ and so $0^{\sigma^{-1}} = 0$. For every $x, y \in T$ we set

$$f(x,y) = ((x+y)^\sigma - x^\sigma - y^\sigma)^{\sigma^{-1}},$$

we see that $f(x, 0) = f(0, x) = 0$ for all $x \in \mathcal{T}$. It is easy to check that σ^{-1} is also a n-multiplicative isomorphism. Thus, for any $u_1, \ldots, u_{n-1} \in T$, we have

$$\begin{aligned} f(x,y)u_1 \cdots u_{n-1} &= ((x+y)^\sigma - x^\sigma - y^\sigma)^{\sigma^{-1}} (u_1^\sigma)^{\sigma^{-1}} \cdots (u_{n-1}^\sigma)^{\sigma^{-1}} \\ &= (((x+y)^\sigma - x^\sigma - y^\sigma) u_1^\sigma \cdots u_{n-1}^\sigma)^{\sigma^{-1}} \\ &= f(xu_1 \cdots u_{n-1}, yu_1 \cdots u_{n-1}). \end{aligned}$$

Similarly we have

$$u_1 \cdots u_{n-1} f(x,y) = f(u_1 \cdots u_{n-1}x, u_1 \cdots u_{n-1}y).$$

The conditions of Corollary 2.1 are met. Hence $f = 0$. That is, $(x+y)^\sigma = x^\sigma + y^\sigma$ for all $x, y \in T$. □

In particular, we have

Corollary 3.1. *Let $T = Tri(A, M, B)$ be a triangular ring such that*

(*i*) *For $a \in A$, if $aA = 0$, then $a = 0$,*
(*ii*) *For $b \in B$, if $Bb = 0$, then $b = 0$.*

Then every multiplicative isomorphism of T onto a ring R is additive.

We remark that Corollary 3.1 improves [2, Theorem 2.3].

Theorem 3.2. *Let $T = Tri(A, M, B)$ be a triangular ring such that*

(*i*) *For $a \in A$, if $aA = 0$, then $a = 0$;*
(*ii*) *For $b \in B$, if $Bb = 0$, then $b = 0$.*

Then any n-multiplicative derivation d of T is additive.

Proof. For any $x, y \in T$, we set

$$f(x, y) = (x + y)^d - x^d - y^d.$$

It is easy to check that the conditions of Corollary 2.1 are met. So the result follows from Corollary 2.1. $\square$

In particular, we have

Corollary 3.2. *Let $T = Tri(A, M, B)$ be a triangular ring such that*

(*i*) *For $a \in A$, if $aA = 0$, then $a = 0$;*
(*ii*) *For $b \in B$, if $Bb = 0$, then $b = 0$.*

Then every multiplicative derivation of T is additive.

Let (M, M^*) be a n-elementary surjective map of $T \times R$. We shall discuss the additivity of M and M^*.

Lemma 3.1. *Let $T = Tri(A, M, B)$ be a triangular ring such that*

(*i*) *For $a \in A$, if $aA = 0$, then $a = 0$;*
(*ii*) *For $b \in B$, if $Bb = 0$, then $b = 0$.*

Suppose that the pair (M, M^*) is a n-elementary surjective map of $T \times R$. Then

(*a*) *$M(0) = 0$ and $M^*(0) = 0$;*
(*b*) *Both M and M^* are bijective;*
(*c*) *The pair $({M^*}^{-1}, M^{-1})$ is a n-elementary map on $T \times R$.*

Proof. We first prove the statement (a). We have

$$M(0) = M(0M^*(0)0 \cdots 0M^*(0)0) = M(0)0M(0) \cdots M(0)0M(0) = 0.$$

Similarly, we have

$$M^*(0) = M^*(0M(0)0 \cdots 0M(0)0) = M^*(0)0M^*(0) \cdots M^*(0)0M^*(0) = 0.$$

We next prove the statement (b). Suppose that $M(a) = M(b)$ for some $a, b \in T$. Taking $y_1 = a$ we get

$$\begin{aligned} &M^*(x_1)aM^*(z_1)\cdots M^*(x_n)y_nM^*(z_n) \\ &\quad = M^*(x_1M(a)z_1 \cdots x_nM(y_n)z_n) \\ &\quad = M^*(x_1M(b)z_1 \cdots x_nM(y_n)z_n) \\ &\quad = M^*(x_1)bM^*(z_1) \cdots M^*(x_n)y_nM^*(z_n). \end{aligned}$$

That is,

$$M^*(x_1)(a-b)M^*(z_1) \cdots M^*(x_n)y_nM^*(z_n) = 0.$$

Since M^* is surjective we get from Proposition 1.2 that $a = b$. So, M is injective.

It remains to show that M^* is injective. Suppose that $M^*(u) = M^*(v)$ for some $u, v \in R$. Since M is surjective there exist $a, b \in T$ such that $u = M(a)$ and $v = M(b)$. Setting $y_1 = u$ we get

$$\begin{aligned} M(x_1)uM(z_1) \cdots M(x_n)y_nM(z_n) &= M(x_1M^*(u)z_1 \cdots x_nM^*(y_n)z_n) \\ &= M(x_1M^*(v)z_1 \cdots x_nM^*(y_n)z_n) \\ &= M(x_1)vM(z_1) \cdots M(x_n)y_nM(z_n). \end{aligned}$$

That is,

$$M(x_1)M(a)M(z_1) \cdots M(x_n)y_nM(z_n) = M(x_1)M(b)M(z_1) \cdots M(x_n)y_nM(z_n).$$

Hence,

$$M^*(M(x_1))(a-b)M^*(M(z_1)) \cdots M^*(M(x_n))M^{-1}(y_n)M^*(M(z_n)) = 0.$$

Since M is bijective and M^* is surjective we get from Proposition 1.2 that $a = b$ and so $u = v$ as desired.

We finally prove the statement (c). That is,

$$\begin{aligned} &{M^*}^{-1}(x_1M^{-1}(y_1)z_1 \cdots x_nM^{-1}(y_n)z_n) \\ &\quad = {M^*}^{-1}(x_1)y_1{M^*}^{-1}(z_1) \cdots {M^*}^{-1}(x_n)y_n{M^*}^{-1}(z_n) \end{aligned}$$

and

$$\begin{aligned} &M^{-1}(y_1{M^*}^{-1}(x_1)u_1 \cdots y_n{M^*}^{-1}(x_n)u_n) \\ &\quad = M^{-1}(y_1)x_1M^{-1}(u_1) \cdots M^{-1}(y_n)x_nM^{-1}(u_n) \end{aligned}$$

for all $x_1, z_1, \ldots, x_n, z_n \in T, y_1, u_1, \ldots, y_n, u_n \in R$. The first identity follows from the following observation.

$$\begin{aligned} &M^*({M^*}^{-1}(x_1)y_1{M^*}^{-1}(z_1) \cdots {M^*}^{-1}(x_n)y_n{M^*}^{-1}(z_n)) \\ &\quad = x_1M^{-1}(y_1)z_1 \cdots x_nM^{-1}(y_n)z_n. \end{aligned}$$

The second one goes similarly. $\square$

Theorem 3.3. *Let $T = Tri(A, M, B)$ be a triangular ring such that*

(*i*) *For $a \in A$, if $aA = 0$, then $a = 0$;*
(*ii*) *For $b \in B$, if $Bb = 0$, then $b = 0$.*

Suppose that the pair (M, M^*) is a n-elementary surjective map of $T \times R$. Then both M and M^* are additive.

Proof. For $x, y \in T$, we set

$$f(x,y) = M^{-1}(M(x+y) - M(x) - M(y)).$$

Since $M(0) = 0$ we easily see that $f(x,0) = 0 = f(0,x)$ for all $x \in T$. For any $x, y, u_1, \ldots, u_{3n-1} \in T$, by assumption we have

$$\begin{aligned} &M(f(xu_1 \cdots u_{3n-1}, yu_1 \cdots u_{3n-1})) \\ &\quad = M((x+y)u_1 \cdots u_{3n-1}) - M(xu_1 \cdots u_{3n-1}) - M(yu_1 \cdots u_{3n-1}) \\ &\quad = (M(x+y) - M(x) - M(y))M^{*^{-1}}(u_1)M(u_2) \cdots M(u_{3n-3})M^{*^{-1}}(u_{3n-2})M(u_{3n-1}) \\ &\quad = M(f(x,y))M^{*^{-1}}(u_1)M(u_2) \cdots M(u_{3n-3})M^{*^{-1}}(u_{3n-2})M(u_{3n-1}) \\ &\quad = M(f(x,y)u_1 \cdots u_{3n-1}). \end{aligned}$$

Since M is bijective we get from the last identity that

$$f(xu_1 \cdots u_{3n-1}, yu_1 \cdots u_{3n-1}) = f(x,y)u_1 \cdots u_{3n-1}.$$

Similarly, we can obtain

$$f(u_1 \cdots u_{3n-1}x, u_1 \cdots u_{3n-1}y) = u_1 \cdots u_{3n-1}f(x,y).$$

The conditions of Corollary 2.1 are met. Therefore $f = 0$ and so M is additive. Since $(M^{*^{-1}}, M^{-1})$ is a n-elementary map of $T \times R$ we get from the above discussion that $M^{*^{-1}}$ is additive and so is M^*. □

In particular, we have

Corollary 3.3. *Let $T = Tri(A, M, B)$ be a triangular ring such that*

(*i*) *For $a \in A$, if $aA = 0$, then $a = 0$,*
(*ii*) *For $b \in B$, if $Bb = 0$, then $b = 0$.*

Suppose that the pair (M, M^*) is an elementary surjective map of $T \times R$. Then both M and M^* are additive.

We remark that Corollary 3.3 improves [2, Theorem 4.1].

Theorem 3.4. *Let $T = Tri(A, M, B)$ be a triangular ring such that, for $b \in B$, $b \circ b_1 = 0$ for all $b_1 \in B$ implies $b = 0$. Suppose that (M, M^*) is an arbitrary Jordan map of $T \times R$, and both M and M^* are surjective. Then both M and M^* are additive.*

Proof. Note that M and M^* are bijective [14, Lemma 2]. By [14, Lemma 3] we know that the pair $(M^{*^{-1}}, M^{-1})$ is a Jordan map of $T \times R$. We set

$$f(x,y) = M^{-1}(M(x+y) - M(x) - M(y)).$$

Note that $M(0) = 0$ and $M^*(0) = 0$ [14, Lemma 1]. Thus, we have

$$f(x,0) = 0 = f(0,x) \quad \text{for all } x \in T.$$

For any $x, y, u \in T$, by assumption we have

$$\begin{aligned} M(f(x \circ u, y \circ u)) &= M((x+y) \circ u) - M(x \circ u) - M(y \circ u) \\ &= (M(x+y) - M(x) - M(y)) \circ M^{*^{-1}}(u) \\ &= M(f(x,y)) \circ M^{*^{-1}}(u) \\ &= M(f(x,y) \circ u). \end{aligned}$$

Hence $f(x \circ u, y \circ u) = f(x, y) \circ u$. The conditions of Corollary 2.2 are met. Therefore $f = 0$ and so M is additive. Since $(M^{*^{-1}}, M^{-1})$ is a Jordan map of $T \times R$ [14, Lemma 3] we get from the above discussion that $M^{*^{-1}}$ is additive and so is M^*. $\square$

We remark that Theorem 3.4 removes the condition (i) in [14, Theorem 1.1].

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